Chromatic Polynomial

For a simple graph G and an integer k, denote by $P(G,k)$ the number of k-colorings of the graph G. We call this function the *chromatic polynomial* of G.

1: For a tree T, show that $P(T,k)$ is really a polynomial.

Solution: It is easy to see verify that for any tree T on n vertices, it is really a polynomial:

$$P(T,k) = k \cdot (k-1)^{n-1}.$$

It can be seen by a greedy coloring.

2: Let G be a graph. What is the smallest k such that $P(G,k) > 0$?

Solution: Observe that $\chi(G)$ is the smallest integer k, for which $P(G,k) > 0$. From definition, it needs to be 0 for smaller k.

The following recursion implies that G is really a polynomial. By G/e we denote the graph obtained by contracting e, i.e. identify the endpoints of e in $G - e$.

Proposition 1. *Let G be a graph and let $e = xy$ be an edge of G. Then,*

$$P(G,k) = P(G-e,k) - P(G/e,k). \tag{1}$$

3: Prove the proposition.

Solution:

Proof. The number of k-colorings of $G - e$, where x and y are colored differently is $P(G,k)$. The number of k-colorings of $G - e$, where x and y are colored the same equals $P(G/e,k)$. From here we get the relation. □

4: Find $P(C_5, x)$ using the above recursion.

Solution: For demonstration let us evaluate the chromatic polynomial of C_5. Notice that $P(C_5,x) = P(P_5,x) - P(C_4,x)$. As P_5 is a tree, we have $P(P_5,x) = x^5 - 4x^4 + 6x^3 - 4x^2 + x$ and with a previous application of the recursion, one can evaluate $P(C_4,x) = x^4 - 4x^3 + 6x^2 - 3x$. And these two give us $P(C_5,x) = x^5 - 5x^4 + 10x^3 - 10x^2 + 4x$.

A *color k-partition* of G is a partition of $V(G)$ on k nonempty disjoint sets

$$V_1, V_2, \ldots, V_k,$$

such that V_i is an independent set in G. Note that a color k-partition of G give us immediately a k-coloring of G with all V_i being its color classes. Denote by $a_k(G)$ the number of color k-partitions of G. Recall that $k_{[i]} = k(k-1)\cdots(k-i+1)$.

Proposition 2. *Let G be a graph on n vertices. Then,*

$$P(G,k) = \sum_{i=1}^{n} a_i(G) k_{[i]}\,. \tag{2}$$

5: Prove the proposition.

Solution:

Proof. If the graph G is properly colored with precisely i colors, then color classes comprise a color i-partition, and their number is $a_i(G)$. As there are k available colors, we can assign colors to the color classes of an i-partition on $k_{[i]}$ ways, which is $a_i(G)k_{[i]}$ all together. For the end observe that every proper coloring can be obtained in this way. $\square$

Proposition 3. *Let G be disjoin union of graphs G_1 and G_2. Then,*

$$P(G,k) = P(G_1,k) \cdot P(G_2,k)\,.$$

6: Prove the above proposition.

Solution: This should be obvious as any k-coloring of G_1 and G_2 give a k-coloring of $G_1 \cup G_2$.

Let G be a union of G_1 and G_2 whose intersection is a clique, i.e.

$$G = G_1 \cup G_2 \quad \text{and} \quad G_1 \cap G_2 = K_r\,.$$

We say G is an r-clique-sum of G_1 and G_2.

Proposition 4. *Let G be a r-clique-sum of graphs grafov G_1 in G_2. Then,*

$$P(G,k) = \frac{P(G_1,k)\cdot P(G_2,k)}{P(K_r,k)}\,.$$

7: Prove the above proposition.

Solution:

Proof. Observe that every k-coloring the complete graph $G_1 \cap G_2$ can be extended to

$$\frac{P(G_i,k)}{k_{[r]}}$$

coloring of G_i za $i = 1,2$. Similarly, it can be extended to

$$\frac{P(G,k)}{k_{[r]}}$$

coloring of G_i. So,

$$\frac{P(G,k)}{k_{[r]}} = \frac{P(G_1,k)}{k_{[r]}} \cdot \frac{P(G_2,k)}{k_{[r]}},$$

and since $P(K_r,k) = k_{[r]}$, we promptly obatin the desired result. $\square$